

ALMOST SQUARE BANACH SPACES

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ABSTRACT. We single out and study a natural class of Banach spaces – almost square Banach spaces. In an almost square space we can find, given a finite set x_1, x_2, \dots, x_N in the unit sphere, a unit vector y such that $\|x_i - y\|$ is almost one. These spaces have duals that are octahedral and finite convex combinations of slices of the unit ball of an almost square space have diameter 2. We provide several examples and characterizations of almost square spaces. We prove that non-reflexive spaces which are M-ideals in their biduals are almost square.

We show that every separable space containing a copy of c_0 can be renormed to be almost square. A local and a weak version of almost square spaces are also studied.

1. INTRODUCTION

Let X be a Banach space with unit ball B_X , unit sphere S_X , and dual space X^* .

Definition 1.1. We will say that a Banach space X is

- (i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \rightarrow 1$ and $\|y_n\| \rightarrow 1$.
- (ii) *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x \pm y_n\| \rightarrow 1$, $\|y_n\| \rightarrow 1$ and $y_n \rightarrow 0$ weakly.
- (iii) *almost square* (ASQ) if for every finite subset $(x_i)_{i=1}^N \subset S_X$ there exists a sequence $(y_n) \subset B_X$ such that $\|x_i \pm y_n\| \rightarrow 1$ for every $i = 1, 2, \dots, N$ and $\|y_n\| \rightarrow 1$.

Obviously WASQ implies LASQ, but it is not completely obvious that ASQ implies WASQ. This will be shown in Theorem 2.8. In the language of Schäffer [39, p. 31] a Banach space X is LASQ if and only if no $x \in S_X$ is *uniformly non-square* (see also [28, Proposition 2.2]).

The above definition was inspired by the following characterizations of octahedral norms shown by Haller, Langemets, and Pöldvere in [21] (see Proposition 2.1, Proposition 2.4, and Lemma 3.1).

Proposition 1.2. *A Banach space X is said to be*

- (i) *locally octahedral if for every $x \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \geq 2 - \varepsilon$.*

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- (ii) weakly octahedral if for every finite subset $(x_i)_{i=1}^N \subset S_X$, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + ty\| \geq (1 - \varepsilon)(|x^*(x_i)| + t)$ for all $i = 1, 2, \dots, N$ and $t > 0$.
- (iii) octahedral if for every finite subset $(x_i)_{i=1}^N \subset S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| \geq 2 - \varepsilon$ for all $i = 1, 2, \dots, N$.

Our main interest in the properties LASQ, WASQ, and ASQ come from their relation to diameter two properties. Let X be a Banach space. Recall that a *slice* of B_X is a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. According to [2] X has the *local diameter 2 property (LD2P)* if every slice of B_X has diameter 2, X has the *diameter 2 property (D2P)* if every nonempty relatively weakly open subset of B_X has diameter 2, and X has the *strong diameter 2 property (SD2P)* if every finite convex combination of slices of B_X has diameter 2. The following theorem is shown in [21] (see Theorems 2.3, 2.7, and 3.3).

Theorem 1.3. *Let X be a Banach space. Then*

- (i) *X has the LD2P if and only if X^* is locally octahedral.*
- (ii) *X has the D2P if and only if X^* is weakly octahedral.*
- (iii) *X has the SD2P if and only if X^* is octahedral.*

The connection between the SD2P and octahedrality has also been studied in [9].

The starting point of this paper was the observation by Kubiak that if X is LASQ then X has the LD2P and similarly if X is WASQ then X has the D2P (see Propositions 2.5 and 2.6 in [28]). The basic idea from Kubiak's proof can be used with a result from [2] to show that ASQ spaces have the SD2P, but we will give a shorter self-contained proof that X^* is octahedral whenever X is ASQ in Proposition 2.5.

It is known that the three diameter 2 properties are different (see [8], [4], and [20]). A natural question is whether LASQ, WASQ, and ASQ are different properties. A consequence of Example 3.3 is that $L_1[0, 1]$ is a WASQ space which is not ASQ.

We now give a short outline of the paper. Section 2 starts with a few characterizations of LASQ and ASQ. In Lemma 2.6 we show that ASQ spaces have to contain almost isometric copies of c_0 . This in turn is used to prove Theorem 2.8, which shows that ASQ implies WASQ. The final result in this section is Theorem 2.9, where we show that every separable Banach space that contains a copy of c_0 can be equivalently renormed to be ASQ.

In Section 3 we will give examples of spaces which are LASQ, WASQ, and ASQ.

In Section 4 we show that non-reflexive spaces which are M-ideals in their biduals are ASQ (see Theorem 4.2). However, the class of ASQ spaces is much bigger than the class of spaces that are M-ideals in their biduals (see the discussion following Corollary 4.3 and Example 6.4).

In Section 5 we study the stability of both (local/weak) octahedrality and (L/W)ASQ when forming absolute sums of Banach spaces. We show that local and weak octahedral, LASQ, and WASQ spaces have nice stability properties (see Propositions 5.2, 5.3, and 5.4) but the situation is different

for ASQ. For $1 \leq p < \infty$ the ℓ_p -sum of two Banach spaces is never ASQ. By Proposition 3.10 in [21] an ℓ_p -sum of two Banach spaces can only be octahedral if $p = 1$ or $p = \infty$.

In Section 6 we connect ASQ with the intersection property of Behrends and Harmand. We show that ASQ spaces fail the intersection property and give a quantitative version of this fact in Proposition 6.1. We also give an example of a space that fails the intersection property and is not LASQ.

We follow standard Banach space notation as used in e.g. [5]. We consider real Banach spaces only.

2. CHARACTERIZATIONS

The following characterization of LASQ and ASQ is clear from the definition. Note that since we have finitely many vectors to play with in the definition of ASQ we may drop the plus-minus sign.

Proposition 2.1. *Let X be a Banach space.*

X is LASQ if and only if for every $x \in S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$.

X is ASQ if and only if for every finite subset $(x_i)_{i=1}^N \subset S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i - y\| \leq 1 + \varepsilon$.

The following lemma is surely well-known but we include it for easy reference.

Lemma 2.2. *Assume $x, y \in S_X$ such that $\|x \pm y\| \leq 1 + \varepsilon$, then*

$$(1 - \varepsilon) \max(|\alpha|, |\beta|) \leq \|\alpha x + \beta y\| \leq (1 + \varepsilon) \max(|\alpha|, |\beta|)$$

for all scalars α and β .

Proof. Let $M = \max(|\alpha|, |\beta|)$. We need to show that

$$1 - \varepsilon \leq \left\| \frac{\alpha}{M}x + \frac{\beta}{M}y \right\| \leq 1 + \varepsilon.$$

It is enough to show

$$1 - \varepsilon \leq \|\lambda x \pm y\| \leq 1 + \varepsilon$$

for all $0 < \lambda \leq 1$. By the triangle inequality, we have that $1 - \varepsilon \leq \|x \pm y\| \leq 1 + \varepsilon$ whenever $x, y \in S_X$ satisfies $\|x \pm y\| \leq 1 + \varepsilon$. It follows that

$$\|\lambda^{-1}y + x\| = \|(1 + \lambda^{-1})y - (y - x)\| \geq (1 + \lambda^{-1}) - \|x - y\| \geq \lambda^{-1} - \varepsilon$$

since $\|x - y\| \leq 1 + \varepsilon$. Hence $\|\lambda x + y\| \geq 1 - \varepsilon \lambda \geq 1 - \varepsilon$.

Also

$$\|\lambda^{-1}y + x\| = \|(\lambda^{-1} - 1)y + (y + x)\| \leq (\lambda^{-1} - 1) + 1 + \varepsilon = \lambda^{-1} + \varepsilon$$

and hence $\|\lambda x + y\| \leq 1 + \varepsilon \lambda \leq 1 + \varepsilon$. \square

Corollary 2.3. *If X is LASQ, then X contains almost isometric copies of ℓ_∞^2 .*

For ASQ Banach spaces we can say even more.

Theorem 2.4. *Let X be a Banach space. If X is ASQ then for every finite dimensional subspace $E \subset X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that*

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E$.

Moreover, given a finite dimensional subspace $F \subset X^*$ we may choose the above y so that $|f(y)| < \varepsilon\|f\|$ for every $f \in F$.

It is clear from Proposition 2.1 that the above theorem is actually a characterization of ASQ.

Proof. Let E be a finite dimensional subspace of X and let $\varepsilon > 0$. Find an $\varepsilon/2$ -net $(x_i)_{i=1}^N$ for S_E . Choose $y \in S_X$ such that $\|x_i \pm y\| \leq 1 + \varepsilon/2$. By the triangle inequality $\|2a\| - \|a - b\| \leq \|a + b\|$ and hence $\|x_i \pm y\| \geq 1 - \varepsilon/2$.

Let $x \in S_E$. Find i such that $\|x_i - x\| \leq \varepsilon/2$. Then

$$\|x \pm y\| \leq \|x_i \pm y\| + \|x - x_i\| \leq 1 + \varepsilon$$

and thus also $\|x \pm y\| \geq 1 - \varepsilon$. Hence by using Lemma 2.2 we get

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E$.

For the moreover part let $F \subset X^*$ be a finite dimensional subspace and let $(f_i)_{i=1}^M \subset S_F$ be an $\varepsilon/2$ -net. For each i choose $z_i \in S_X$ with $f_i(z_i) > 1 - \varepsilon/4$. Let $E' = \text{span}\{E, (z_i)_{i=1}^M\}$ and use the first part of the proof to find $y \in S_X$ such that

$$(1 - \varepsilon/4) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon/4) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E'$.

Since $|f_i(z_i \pm y)| \leq \|z_i \pm y\| \leq 1 + \varepsilon/4$ we get

$$\begin{aligned} -\varepsilon/2 &= 1 - \varepsilon/4 - (1 + \varepsilon/4) \leq f_i(z_i) - f_i(z_i - y) = f_i(y) \\ &\leq f_i(z_i + y) - f_i(z_i) \leq 1 + \varepsilon/4 - 1 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

so that $|f_i(y)| < \varepsilon/2$. Thus for every $f \in S_F$, and for some i , we have $|f(y)| \leq |(f - f_i)(y)| + |f_i(y)| \leq \varepsilon$. \square

Proposition 2.5. *If X is ASQ, then X^* is octahedral.*

Proof. Let $x_1^*, x_2^*, \dots, x_n^* \in S_{X^*}$ and $\varepsilon > 0$.

Find $x_1, x_2, \dots, x_n \in S_X$ such that $x_i^*(x_i) > 1 - \varepsilon$. Using Theorem 2.4 find $y \in S_X$ such that $\|x_i \pm y\| \leq 1 + \varepsilon$ and $|x_i^*(y)| < \varepsilon$.

Find $y^* \in S_{X^*}$ such that $y^*(y) = 1$. Then

$$1 + \varepsilon \geq \|x_i \pm y\| \geq \pm y^*(x_i) + y^*(y) = \pm y^*(x_i) + 1$$

and thus $|y^*(x_i)| \leq \varepsilon$. Now

$$\begin{aligned} \|x_i + y\| \|x_i^* + y^*\| &\geq x_i^*(x_i) + x_i^*(y) + y^*(x_i) + y^*(y) \\ &> 1 - \varepsilon - 2\varepsilon + 1 \end{aligned}$$

and hence

$$\|x_i^* + y^*\| > \frac{2 - 3\varepsilon}{1 + \varepsilon}$$

which shows that X^* is octahedral by Proposition 2.1 in [21]. \square

Repeated use of Theorem 2.4 gives the following lemma.

Lemma 2.6. *If X is ASQ, then for every finite dimensional subspace E of X and every $\varepsilon > 0$ there exists a subspace Y of X which is ε -isometric to c_0 such that $E \oplus Y$ is ε -isometric to $E \oplus_\infty c_0$.*

Proof. Find sequence $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\prod_{n=1}^{\infty} (1 + \varepsilon_n) < 1 + \varepsilon$ and $\prod_{n=1}^{\infty} (1 - \varepsilon_n) > 1 - \varepsilon$. Using Theorem 2.4 we inductively choose a sequence $(y_n) \subset S_X$ such that

$$(1 - \varepsilon_n) \max\{\|e\|, |\lambda|\} \leq \|e + \lambda y_n\| \leq (1 + \varepsilon_n) \max\{\|e\|, |\lambda|\}$$

for every $e \in \text{span}\{E, (y_i)_{i=1}^{n-1}\}$ and every $\lambda \in \mathbb{R}$. Then $Y = \overline{\text{span}\{(y_n)\}}$ is ε -isometric to c_0 and defining $S : E \oplus_{\infty} c_0 \rightarrow E \oplus Y$ by $S(e, a) = e + Ta$ where $T : c_0 \rightarrow Y$ is the ε -isometry. We have

$$\begin{aligned} \|S(e, \sum_{n=1}^N a_n e_n)\| &= \|e + \sum_{n=1}^N a_n y_n\| \leq (1 + \varepsilon_N) \max\{\|e + \sum_{n=1}^{N-1} a_n y_n\|, |a_N|\} \\ &\leq \cdots \leq \prod_{n=1}^N (1 + \varepsilon_n) \max\{\|e\|, |a_1|, |a_2|, \dots, |a_N|\} \\ &< (1 + \varepsilon) \|(e, \sum_{n=1}^N a_n e_n)\|, \end{aligned}$$

and similarly $\|S(e, \sum_{n=1}^N a_n e_n)\| > (1 - \varepsilon) \|(e, \sum_{n=1}^N a_n e_n)\|$. Thus S must be an ε -isometry onto $E \oplus Y$ since T is onto Y . \square

Remark 2.7. In Proposition 6 in [38] Pfitzner showed that M -embedded spaces contain an asymptotically isometric copy of c_0 using the local characterization of M -ideals. If X is ASQ and we use Theorem 2.4 in Pfitzner's proof we get that X actually contains an asymptotically isometric copy of c_0 , and hence X^* contains an asymptotically isometric copy of ℓ_1 (see Theorem 2 in [14]).

A consequence of Lemma 2.6 is that the sequence (y_n) in the definition of ASQ may be chosen to be weakly null. This enables us to connect the ASQ and WASQ properties.

Theorem 2.8. *If a Banach space X is ASQ then for every $x_1, x_2, \dots, x_N \in S_X$ there exists $(y_n) \subset B_X$ such that $\|x_i \pm y_n\| \rightarrow 1$ for all i , $y_n \rightarrow 0$ weakly, and $\|y_n\| \rightarrow 1$.*

In particular, ASQ implies WASQ.

Proof. Let $x_1, x_2, \dots, x_N \in S_X$ and $E = \text{span}\{(x_i)_{i=1}^N\}$, and choose a sequence $(y_n) \subset S_X$ as in the proof of Lemma 2.6. Let $Z = E \oplus_{\infty} c_0$ and $z_i = (x_i, 0) \in Z$. Since the standard basis $(e_n)_{n=1}^{\infty} \subset S_{c_0}$ is weakly null so is $w_n = (0, e_n)$ in Z . By Lemma 2.6 there exists an ε -isometry S from Z onto $E \oplus Y$ where $Y = \overline{\text{span}\{(y_n)\}}$. The weak-weak continuity of S shows that $y_n \rightarrow 0$ weakly in $E \oplus Y$ and hence also in X .

By definition $S(e, \pm e_n) = e \pm y_n$ for every $e \in E$. Since

$$(1 - \varepsilon_n) \max\{\|e\|, 1\} \leq \|e \pm y_n\| \leq (1 + \varepsilon_n) \max\{\|e\|, 1\}$$

for every $e \in E$, we in particular have $(1 - \varepsilon_n) \leq \|x_i \pm y_n\| \leq (1 + \varepsilon_n)$, so $\|x_i \pm y_n\| \rightarrow 1$. \square

We know that every ASQ space contains c_0 . Next we will show that a separable Banach space containing c_0 can be equivalently renormed to be ASQ. For separable spaces this improves [2, Proposition 4.7] which says that

any Banach space containing c_0 can be equivalently renormed to have the SD2P (see also [35, Proposition 2.6] for the D2P case).

The proof of the following result is based on a renorming technique which appears in [8, Lemma 2.3].

Theorem 2.9. *A separable Banach space can be equivalently renormed to be ASQ if and only if it contains a copy of c_0 .*

Proof. As an ASQ-space contains c_0 , the “only if” part is clear.

For the “if” part, first renorm X to contain c_0 isometrically [11, Lemma 8.1]. Denote by $\|\cdot\|$ the new norm on X . By Sobczyk’s theorem there exists a projection P onto c_0 with $\|P\| \leq 2$. Define

$$\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\}.$$

Then $\|\cdot\|$ is a norm on X which satisfies $\frac{1}{2}\|x\| \leq \|x\| \leq 3\|x\|$. Also $\|\cdot\|$ extends the max norm $\|\cdot\|$ on c_0 and we get that c_0 is an M-summand in X and hence X is ASQ. (See the proof of Proposition 5.7.) \square

It is clear from the proof that all Banach spaces containing a complemented copy of c_0 can be renormed to be ASQ. We do not know whether or not the same is true for a general Banach space.

3. EXAMPLES

In this section we will provide examples of Banach spaces which are LASQ, WASQ, and ASQ and spaces which are not.

By considering the constant 1 function in ℓ_∞ , $C(K)$, or $L_\infty[0, 1]$ it is clear that neither of these spaces are LASQ. It is also obvious that c_0 is ASQ. In fact, we have the following.

Example 3.1. Given a sequence of Banach spaces (X_i) the c_0 -sum $c_0(X_i)$ is both WASQ and ASQ.

Let $(x_i)_{i=1}^N \subset S_{c_0(X_i)}$. By choosing j_n large enough we may assume that $\|x_i(j_n)\| < \frac{1}{n}$ for all $i = 1, 2, \dots, N$. Choosing $y_{j_n} \in S_{X_{j_n}}$ and defining $y_n \in c_0(X_i)$ by $y_n(j) = 0$ for all j except $y_n(j_n) = y_{j_n}$ we see that $\|x_i \pm y_n\| \rightarrow 1$ as $n \rightarrow \infty$.

As an extreme example $c_0(L_1[0, 1])$ is ASQ and octahedral (and even has the Daugavet property).

Example 3.2. A separable Banach space X has Kalton and Werner’s *property* (m_∞) if

$$\limsup_n \|x + y_n\| = \max(\|x\|, \limsup_n \|y_n\|)$$

for every $x \in X$ whenever $y_n \rightarrow 0$ weakly. If X has property (m_∞) then X is ASQ if and only if X lacks the Schur property.

However, if X does not contain a copy of ℓ_1 , then by Theorem 3.5 in [26] X has property (m_∞) if and only if X is almost isometric to a subspace of c_0 . This is much stronger than ASQ, see Corollary 4.3 below.

In [17] Gao and Lau considered the following parameter

$$G(X) = \sup\{\inf\{\max\{\|x + y\|, \|x - y\|\}, y \in S_X\}, x \in S_Y\}.$$

We see from Proposition 2.1 that X is LASQ if and only if $G(X) = 1$. Gao and Lau showed that $L_1[0, 1]$ is LASQ while $L_p[0, 1]$, $1 < p \leq \infty$, and ℓ_p ,

$1 \leq p \leq \infty$, are not. In [40] Whitley introduced the *thinness index*, $t(X)$, of a Banach space X . It is not difficult to see that Whitley's original definition of $t(X)$ is equivalent to

$$t(X) = \sup_{x_1, x_2, \dots, x_N \in S_X} \inf_{y \in S_X} \max_i \|x_i - y\|.$$

We see that $t(X) = 1$ if and only if X is ASQ.

As noted above Gao and Lau have shown that $L_1[0, 1]$ is LASQ, in fact it is WASQ but it is not ASQ. This is a special case of our next example which concerns Cesàro function spaces. We will need a bit of Banach lattice notation (for more see e.g. [34]).

For an interval $I \subset \mathbb{R}$ by $L_0(I)$ we denote the set of all (equivalence classes of) real valued Lebesgue measurable (finite almost everywhere) functions on I . A *Banach function lattice* is a Banach space $E = E(I) \subset L_0(I)$ such that if $|f(x)| \leq |g(x)|$ a.e. with $f \in L_0(I)$ and $g \in E$, then $f \in E$ and $\|f\| \leq \|g\|$. E is *order continuous* if for every $f \in E$ and every $0 \leq f_n \leq |f|$ a.e. such that $f_n \downarrow 0$ a.e. we have that $\|f_n\| \downarrow 0$. E has the *Fatou property* if for any sequence $(f_n) \subset E$ and any $f \in L_0(I)$ such that $0 \leq f_n \leq f$ a.e., $f_n \uparrow f$ a.e., and $\sup_n \|f_n\| < \infty$ we have that $f \in E$ and $\|f\| = \lim_n \|f_n\|$.

For $I = (0, l)$ with $0 < l \leq \infty$ fixed and a fixed weight $0 < \omega \in L_0(I)$ we can define a norm on $L_0(I)$ for $1 \leq p < \infty$ by

$$\|f\|_{C_{p,\omega}} = \left(\int_I (\omega(x) \int_0^x |f(t)| dt)^p dx \right)^{1/p}$$

The *weighted Cesàro function space* on I is defined by

$$C_{p,\omega} = C_{p,\omega}(I) = \{f \in L_0(I) : \|f\|_{p,\omega} < \infty\}.$$

It is known that $C_{p,\omega}$ in the natural pointwise order is a separable order continuous Banach function lattice with the Fatou-property (see [27, Lemma 3.1]). Note that the space $C_{1,1/x}[0, 1]$ is isometrically isomorphic to $L_1[0, 1]$ (see e.g. [7, p. 4293]).

Example 3.3. The Cesàro function space $C_{p,\omega}$ is WASQ but not ASQ.

Kubiak [28, Lemma 3.3] proved that $C_{p,\omega}$ is WASQ. In Proposition 3.5 below we will show that $C_{p,\omega}$ does not contain c_0 so by Lemma 2.6 it is not ASQ.

Theorem 2.8 and the example above shows that ASQ is strictly stronger than WASQ.

Question 3.4. *Is WASQ strictly stronger than LASQ?*

Proposition 3.5. *The space $C_{p,\omega}$ does not contain an isomorphic copy of c_0 .*

Proof. Let (f_n) be an increasing norm bounded sequence in $C_{p,\omega}$. By [34, Theorem 1.c.4] it is enough to show that (f_n) has a norm limit. If (f_n) has a pointwise a.e. limit f , then it follows from the Fatou property that f is in $C_{p,\omega}$. Let $g_n = f - f_n$. Then $0 \leq g_n \leq f - f_1$ and $g_n \downarrow 0$. By order continuity we get that $\|f - f_n\| = \|g_n\| \rightarrow 0$ as wanted.

It only remains to prove that the pointwise limit exists. (f_n) increasing means that $f_n(x) \leq f_{n+1}(x)$ for a.e. x . By completeness it is enough to show that $(f_n(x))$ is a bounded sequence for a.e. x . Assume not, i.e. that

$\sup_n f_n(x) = \infty$ on a compact A of positive Lebesgue measure $\lambda(A) > 0$. Split A into two parts A_1 and A_2 with $\lambda(A_1) > 0$ and $\lambda(A_2) > 0$ such that $x \leq y$ for all $x \in A_1$ and $y \in A_2$.

We know that

$$K = \int_{A_2} w(x)^p dx > 0.$$

Let $S = \sup_n \|f_n\| < \infty$. Choose k such that $S^p < M^p K$ where

$$M = \int_{A_1} |f_k(t)| dt.$$

Then

$$\begin{aligned} S^p &\geq \|f_k\|^p = \int_I \left(w(x) \int_0^x |f_k(t)| dt \right)^p dx \geq \int_{A_2} \left(w(x) \int_0^x |f_k(t)| dt \right)^p dx \\ &\geq \int_{A_2} \left(w(x) \int_{A_1} |f_k(t)| dt \right)^p dx = \int_{A_2} (w(x)M)^p dx = M^p K \end{aligned}$$

and we have a contradiction. \square

Let us end this section by providing examples of ASQ, LASQ, and non-LASQ from the class of Lindenstrauss spaces (i.e. the Banach spaces with duals isometric to $L_1(\mu)$ for some positive measure μ). To see that the examples below are Lindenstrauss cf. e.g. [32, p. 80]. From Theorem 6.1 (14) in [32] we see that only Lindenstrauss spaces without extreme points can be LASQ.

Recall the thinness index $t(X)$ from the discussion following Example 3.2. In [40, Lemma 8] Whitley showed that for any compact Hausdorff we have $t(C(K)) = 2$, while $t(C_0(K)) = 1$ if K locally compact. Hence $C(K)$ -spaces are not ASQ, while $C_0(K)$ -space are. The idea is that every finite set of functions need a common zero for the space to be ASQ. Another example of the same idea is the following.

Example 3.6. Let K be a compact Hausdorff space and $\sigma : K \rightarrow K$ involutory homeomorphism (i.e. $\sigma^2 = id_K$) with a non-isolated fixed point. Then $X = \{f \in C(K) : f(x) = -f(\sigma(x)) \text{ for all } x \in K\}$ is ASQ.

If x_0 is a fixed point for σ , then $f(x_0) = -f(\sigma(x_0)) = -f(x_0)$ for all $f \in X$. Hence given $f_1, f_2, \dots, f_N \in S_X$ and $\varepsilon > 0$ there is a neighborhood U of x_0 in K where $|f_i(x)| < \varepsilon$ for all $x \in U$ and $i = 1, 2, \dots, N$. If we let $g \in S_X$ have support on $U \cup \sigma(U)$ then $\|f_i \pm g\| \leq 1 + \varepsilon$ and thus X is ASQ.

It is also not difficult to find LASQ subspaces of $C(K)$ -spaces.

Example 3.7. Let $X = \{f \in C[0, 1] : f(0) = -f(1)\}$. Then X is LASQ but not ASQ.

Since $f \in S_X$ has a zero in $[0, 1]$ we can always find an interval where $|f(x)|$ is as small as we like. Any $g \in S_X$ with support in this interval has $\|f \pm g\|$ small so X is LASQ.

To see that X is not ASQ let f_1 be any function in S_X which is equal to 1 on $[0, \frac{1}{2}]$ and let $f_2(x) = f_1(1 - x)$. Then $\max_i \|f_i \pm g\| = 2$ for any $g \in S_X$.

Remark 3.8. It is clear that in the above LASQ and ASQ examples we can construct a bounded sequence (g_n) in the subspace X of $C(K)$ such that $g_n \rightarrow 0$ pointwise in $C(K)$. Thus $g_n \rightarrow 0$ weakly in $C(K)$, see e.g. Theorem 1

in [13, p. 66], and hence must also be weakly null in X . It follows that these examples are also WASQ.

4. M-EMBEDDED SPACES

In this section we will show that all M-embedded spaces are ASQ. We start by recalling some definitions.

A subspace Y of a Banach space X is an *ideal* in X if the annihilator Y^\perp is the kernel of a norm one projection on X^* . The subspace Y is called *locally 1-complemented* in X if for every finite dimensional subspace E of X and every $\varepsilon > 0$ there exists a linear operator $u : E \rightarrow Y$ such that $u(e) = e$ for all $e \in E \cap Y$ and $\|u\| \leq 1 + \varepsilon$. Fakhoury [16, Théorème 2.14] proved that Y is an ideal in X precisely when it is locally 1-complemented in X .

Following [3] we say that Y is an *almost isometric ideal* (ai-ideal) in X if Y is locally 1-complemented in X in such a way that the operator $u : E \rightarrow Y$ is an almost isometry, i.e. in addition to the above we have $(1 + \varepsilon)^{-1}\|e\| \leq \|u(e)\| \leq (1 + \varepsilon)\|e\|$ for all $e \in E$. The fact that every Banach space is an ai-ideal in its bidual is commonly referred to as *the Principle of Local Reflexivity* (PLR).

Lemma 4.1. *If X is (L)ASQ and Y is an ai-ideal in X then Y is (L)ASQ.*

Proof. We only show the ASQ case. Let $y_1, y_2, \dots, y_N \in S_Y$ and $0 < \varepsilon < 1$. Find $x \in S_X$ such that $\|y_i - x\| \leq 1 + \varepsilon/4$. Now, choose an $\varepsilon/4$ -isometry $u : E \rightarrow Y$ such that u is the identity on $E \cap Y$ where $E = \text{span}\{(y_j)_{j=1}^N, x\}$. Define $z = u(x)/\|u(x)\|$. Then $z \in S_Y$ and $\|u(x) - z\| = |\|u(x)\| - 1| \leq \varepsilon/4$ and

$$\|y_i - z\| \leq \|u(y_i - x)\| + \|u(x) - z\| \leq (1 + \frac{\varepsilon}{4})(1 + \frac{\varepsilon}{4}) + \frac{\varepsilon}{4} \leq 1 + \varepsilon.$$

Thus Y is ASQ by Proposition 2.1. \square

If Y is an ideal in X with an ideal projection P on X^* which for every $x^* \in X^*$ satisfies $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$, then Y is said to be an *M-ideal* in X (P is called the M-ideal projection on X^*). If X is an M-ideal in X^{**} , then X is said to be *M-embedded*. For M-ideals we often get ASQ for free.

Theorem 4.2. *Let Y be a proper subspace of a non-reflexive Banach space X . If Y is both an M-ideal and an ai-ideal in X , then Y is ASQ.*

Proof. Let $\varepsilon > 0$ and choose $0 < \delta < 1$ with $(1 + \delta)^2(1 + 3\delta(1 + \delta)^2) < 1 + \varepsilon$. Write $X^{**} = (PX^*)^\perp \oplus_\infty Y^{\perp\perp}$. This is possible as Y is an M-ideal in X and thus $X^* = P(X^*) \oplus_1 Y^\perp$ (P denotes here the M-ideal projection on X^*). Let $y_1, y_2, \dots, y_N \in S_Y$ and $z \in S_{(PX^*)^\perp}$, and put $E = \text{span}\{(y_i)_{i=1}^N, z\} \subset X^{**}$. Use the PLR to find a δ -isometry $v : E \rightarrow X$ which is the identity on $E \cap X$. Further, put $F = v(E) \subset X$ and use that Y is an ai-ideal in X to find a δ -isometry $u : F \rightarrow Y$ which is the identity on $F \cap Y$. Now with $y = uv(z)/\|uv(z)\| \in S_Y$ we use $uv(y_i) = y_i$ to get

$$\begin{aligned} \|y_i - y\| &= \|y_i - \frac{uv(z)}{\|uv(z)\|}\| \leq (1 + \delta)^2 \|y_i - \frac{z}{\|uv(z)\|}\| \\ &\leq (1 + \delta)^2 (\|y_i - z\| + \|z - \frac{z}{\|uv(z)\|}\|) < 1 + \varepsilon \end{aligned}$$

since

$$\begin{aligned} \left\| z - \frac{z}{\|uv(z)\|} \right\| &= \frac{1}{\|uv(z)\|} |1 - \|uv(z)\|| \\ &\leq (1 + \delta)^2 (|1 - \|v(z)\|| + |\|v(z)\| - \|uv(z)\||) \\ &\leq (1 + \delta)^2 (\delta + \delta(1 + \delta)) \leq 3\delta(1 + \delta)^2. \end{aligned}$$

Using Proposition 2.1 we are done. \square

Since every Banach space is an ai-ideal in its bidual by the PLR we immediately have the following corollary.

Corollary 4.3. *Non-reflexive M-embedded spaces are ASQ.*

The following spaces are examples of M-embedded spaces: $c_0(\Gamma)$ (for any set Γ), $\mathcal{K}(H)$ of compact operators on a Hilbert space H , and $C(\mathbb{T})/A$ where \mathbb{T} denotes the unit circle and A the disk algebra. (For more examples see Chapter III in [25].) From Example 3.1 the space $c_0(\ell_1)$ is ASQ. However, this space contains a copy of ℓ_1 and therefore can not be M-embedded ([23, Theorems 3.4.a and 3.5]). Thus the class of ASQ spaces properly contains the class of M-embedded spaces.

5. STABILITY

We start this section by recalling the notion of an absolute sum of a family of Banach spaces. Our goal is to show that LASQ and WASQ spaces are stable under absolute sums (see Propositions 5.2 and 5.3). It turns out that locally and weakly octahedral Banach spaces are stable by forming absolute sums too (see Propositions 5.3 and 5.4).

Definition 5.1. Let I be a non-empty set and let E be a \mathbb{R} -linear subspace of \mathbb{R}^I . An *absolute norm* on E is a complete norm $\|\cdot\|_E$ satisfying

- (i) Given $(a_i)_{i \in I}, (b_i)_{i \in I} \in \mathbb{R}^I$ with $|b_i| \leq |a_i|$ for every $i \in I$, if $(a_i)_{i \in I} \in E$, then $(b_i)_{i \in I} \in E$ with $\|(b_i)_{i \in I}\|_E \leq \|(a_i)_{i \in I}\|_E$.
- (ii) For every $i \in I$, the function $e_i : I \rightarrow \mathbb{R}$ given by $e_i(j) = \delta_{ij}$ for $j \in I$, belongs to E and $\|e_i\|_E = 1$.

Let $E \subset \mathbb{R}^I$ with an absolute norm. Then $\ell_1(I) \subseteq E \subseteq \ell_\infty(I)$, and E can be viewed as a Köthe function space (and hence a Banach lattice) on the space $(I, \mathcal{P}(I), \mu)$, where $\mathcal{P}(I)$ is the power set of I and μ is the counting measure on I . It is known that E is order continuous if and only if E does not contain an isomorphic copy of ℓ_∞ if and only if $\text{span}\{e_i : i \in I\}$ is dense in E .

The Köthe dual E' of a Banach space $E \subset \mathbb{R}^I$ with absolute norm is the linear subspace of \mathbb{R}^I defined by

$$E' := \left\{ (a_i)_{i \in I} \in \mathbb{R}^I : \sup \sum_{i \in I} |a_i b_i| < \infty, (b_i)_{i \in I} \in E \right\}.$$

It is not hard to see that

$$\|(a_i)_{i \in I}\|_{E'} := \sup \left\{ \sum_{i \in I} |a_i b_i| : (b_i)_{i \in I} \in E \right\}$$

defines an absolute norm on E' . Every $(b_i)_{i \in I} \in E'$ defines a functional on E by

$$(a_i)_{i \in I} \rightarrow \sum_{i \in I} b_i a_i.$$

This induces an embedding $E' \rightarrow E^*$ which is easily seen to be linear and isometric. If $\text{span}\{e_i : i \in I\}$ is dense in E then the embedding $E' \rightarrow E^*$ is surjective, and so E' and E^* can be identified.

Now, if $(X_i)_{i \in I}$ is a family of Banach spaces we put

$$[\oplus_{i \in I} X_i]_E := \{(x_i)_{i \in I} \in \prod_{i \in I} X_i : (\|x_i\|)_{i \in I} \in E\}.$$

It is clear that this defines a subspace of the product space $[\oplus_{i \in I} X_i]_E$ which becomes a Banach space when given the norm

$$\|(x_i)_{i \in I}\| := \|(\|x_i\|)_{i \in I}\|_E, \quad (x_i)_{i \in I} \in [\oplus_{i \in I} X_i]_E.$$

This Banach space is said to be the *absolute sum of the family $(X_i)_{i \in I}$ with respect to E* . Every $(x_i^*)_{i \in I} \in [\oplus_{i \in I} X_i^*]_{E'}$ defines a functional on $[\oplus_{i \in I} X_i]_E$ by

$$(x_i)_{i \in I} \rightarrow \sum_{i \in I} x_i^*(x_i).$$

This embedding is isometric and is onto if $\text{span}\{e_i : i \in I\}$ is dense in E .

Putting $I = \mathbb{N}$ and $E = \ell_p(I)$ it is clear that for $1 \leq p \leq \infty$ the ℓ_p sum (c_0 sum if $p = \infty$) of a family of Banach spaces $(X_i)_{i \in I}$ is an absolute sum with respect to E (for which $[\oplus_{i \in I} X_i^*]_{E'} = [\oplus_{i \in I} X_i]_E^*$ as $\text{span}\{e_i : i \in I\}$ is dense in $\ell_p(I)$ in this case). In [21, Propositions 3.4 and 3.7] it was proved that locally and weakly octahedral spaces are stable by taking ℓ_p sums of two Banach spaces. This can also be obtained from Propositions 5.3 and 5.4 below. First we show that WASQ is stable by taking absolute sums.

Proposition 5.2. *Let E be a subspace of \mathbb{R}^I with an absolute norm such that $\text{span}\{e_i : i \in I\}$ is dense in E . If $(X_i)_{i \in I}$ is a family of Banach spaces which are WASQ, then $X = (\oplus_{i \in I} X_i)_E$ is WASQ.*

Proof. Let $x = (x_i)_{i \in I} \in S_X$. Our task is to find a weakly-null sequence $(y_n) \subset S_X$ such that

$$\|x \pm y_n\|_E \rightarrow 1.$$

Since $\text{span}\{e_i : i \in I\}$ is dense in E we may assume that $J = \{i \in I : x_i \neq 0\}$ is finite. By assumption, for every $i \in J$, there exist weakly-null sequences $(y_i^n) \subset S_{X_i}$ such that

$$\left\| \frac{x_i}{\|x_i\|} \pm y_i^n \right\| \leq 1 + n^{-1}.$$

If we let $y_n = (\|x_i\| y_i^n)_{i \in I}$, with $\|x_i\| y_i^n = 0$ for $i \notin J$, then $\|y_n\|_E = 1$ and

$$\|x \pm y_n\|_E = \|(\|x_i \pm \|x_i\| y_i^n\|)_{i \in I}\|_E \leq \|((1 + n^{-1})\|x_i\|)_{i \in I}\|_E \leq 1 + n^{-1}.$$

Let $x^* = (x_i^*)_{i \in I} \in X^*$. Since $x^*(y_n) = \sum_{i \in J} x_i^*(y_i^n)$ and J is finite we get $x^*(y_n) \rightarrow 0$ and thus (y_n) is weakly-null. \square

In the proof above we assumed that our unit vector had finite support in order to get a weakly-null sequence. For absolute sums of LASQ and locally octahedral spaces this is not necessary and the first part of the proof above gives the following proposition.

Proposition 5.3. *Let I be a set, E a subspace of \mathbb{R}^I with an absolute norm, and $(X_i)_{i \in I}$ a family of Banach spaces which are locally octahedral (resp. LASQ). Then their absolute sum $X = (\oplus_{i \in I} X_i)_E$ is locally octahedral (resp. LASQ).*

For absolute sums of weakly octahedral spaces we have to work a bit harder.

Proposition 5.4. *Let I be a set, E a subspace of \mathbb{R}^I with an absolute norm such that $\text{span}\{e_i : i \in I\}$ is dense in E , and $(X_i)_{i \in I}$ a family of Banach spaces which are weakly octahedral. Then their absolute sum $X = (\oplus_{i \in I} X_i)_E$ is weakly octahedral.*

Proof. Let $\varepsilon > 0$, let $x_1 = (x_i^1)_{i \in I}, x_2 = (x_i^2)_{i \in I}, \dots, x_N = (x_i^N)_{i \in I} \in S_X$, and $x^* = (x_i^*)_{i \in I} \in B_{X^*}$. Our task here is to find $y \in S_X$ such that for all $t > 0$ and $k = 1, 2, \dots, N$

$$\|x_k + ty\|_E \geq (1 - \varepsilon)(|\sum_{i \in I} x_i^*(x_i^k)| + t).$$

Let $z_i^* = \frac{x_i^*}{\|x_i^*\|}$ if $x_i^* \neq 0$ and $z_i^* = 0$ otherwise. By the weak octahedrality of X_i , for every $i \in I$, there exists a $y_i \in S_{X_i}$ such that for all $t > 0$ and $k = 1, 2, \dots, N$

$$(1) \quad \left\| \frac{x_i^k}{\|x_i^k\|} + ty_i \right\| \geq (1 - \varepsilon/2) \left(\frac{|z_i^*(x_i^k)|}{\|x_i^k\|} + t \right).$$

If $x_i^k = 0$ for some $i \in I$, then take y_i to be any element from S_{X_i} . Now (1) implies that for all $t > 0$ and $k = 1, 2, \dots, N$

$$\|x_i^k + ty_i\| \geq (1 - \varepsilon/2) \left(\frac{|x_i^*(x_i^k)|}{\|x_i^*\|} + t \right).$$

Since $\|x^*\| = \|(\|x_i^*\|)_{i \in I}\|_{E^*} \leq 1$, there is a list of reals $(\alpha_i)_{i \in I} \subset \mathbb{R}$ such that $\|(\alpha_i)_{i \in I}\|_E = \|(|\alpha_i|)_{i \in I}\|_E = 1$ and

$$\sum_{i \in I} \|x_i^*\| \cdot |\alpha_i| > \|x^*\| \left(1 - \frac{\varepsilon/2}{1 - \varepsilon/2} \right).$$

We take $y = (|\alpha_i|y_i)_{i \in I} \in S_X$ to get

$$\begin{aligned} \|x^*\| \|x_k + ty\|_E &\geq \sum_{i \in I} \|x_i^*\| \cdot \|x_i^k + |\alpha_i|ty_i\| \\ &\geq (1 - \varepsilon/2) \sum_{i \in I} \|x_i^*\| \left(\frac{|x_i^*(x_i^k)|}{\|x_i^k\|} + |\alpha_i|t \right) \\ &\geq (1 - \varepsilon/2) \left(\left| \sum_{i \in I} x_i^*(x_i^k) \right| + \|x^*\| \left(1 - \frac{\varepsilon/2}{1 - \varepsilon/2} \right) t \right) \\ &\geq \left(1 - \frac{\varepsilon/2}{1 - \varepsilon/2} \right) (1 - \varepsilon/2) \|x^*\| \left(\left| \sum_{i \in I} x_i^*(x_i^k) \right| + t \right) \\ &= \|x^*\| (1 - \varepsilon) \left(\left| \sum_{i \in I} x_i^*(x_i^k) \right| + t \right). \end{aligned}$$

Dividing both sides by $\|x^*\|$ we get the desired inequality. \square

We have seen that for a sequence of non-trivial Banach spaces (X_i) the space $c_0(X_i)$ is always ASQ. Similarly $\ell_1(X_i)$ is always octahedral.

Note that $X \oplus_p Y$, $1 < p < \infty$, can never be ASQ, because it fails the SD2P (see [4, Theorem 3.2] or [20, Theorem 1]). But even though the SD2P property is stable by forming ℓ_1 sums (see [2, Theorem 2.7 (iii)]), it turns out that the ℓ_1 sum of Banach spaces can never be ASQ.

Lemma 5.5. *Let X and Y be nontrivial Banach spaces. Then $X \oplus_1 Y$ is never ASQ.*

Proof. Let $Z = X \oplus_1 Y$, $x \in S_X$, and $y \in S_Y$. Consider norm 1 elements $z_1 = (-\frac{1}{3}x, \frac{2}{3}y)$ and $z_2 = (\frac{2}{3}x, -\frac{1}{3}y)$. Assume for contradiction that there is a $w = (w_x, w_y) \in S_Z$ with $\|z_i \pm w\| \leq 1 + \frac{1}{9}$. Then

$$\begin{aligned} \|w_x\| + \|\frac{2}{3}y\| &\leq \frac{1}{2} \left(\left\| -\frac{1}{3}x + w_x \right\| + \left\| \frac{2}{3}y + w_y \right\| + \left\| \frac{1}{3}x + w_x \right\| + \left\| \frac{2}{3}y - w_y \right\| \right) \\ &\leq \max\{\|z_1 + w\|, \|z_1 - w\|\} \leq 1 + \frac{1}{9} \end{aligned}$$

so that $\|w_x\| \leq \frac{1}{3} + \frac{1}{9}$. Similarly $\|w_y\| \leq \frac{1}{3} + \frac{1}{9}$. We get $\|w\| < 1$ which is a contradiction. \square

Corollary 5.6. *Let X and Y be nontrivial Banach spaces and $1 \leq p < \infty$.*

(i) *$X \oplus_p Y$ is LASQ if and only if X and Y are LASQ.*

(ii) *$X \oplus_p Y$ is WASQ if and only if X and Y are WASQ.*

Proof. (i). One direction is Proposition 5.3. Let us show that X is LASQ whenever $X \oplus_p Y$ is.

The function $f(x) = x^{1/p}$ is uniformly continuous on $[0, 2]$ so given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Also the function $g(x) = x^p$ is continuous at $x = 1$ so there exists $\eta > 0$ such that $|g(1) - g(y)| \leq \delta$ whenever $|1 - y| \leq \eta$.

Assume $x \in S_X$. Since $X \oplus_p Y$ is LASQ there exists $(u, v) \in S_{X \oplus_p Y}$ such that

$$\|(x, 0) \pm (u, v)\|_p = (\|x \pm u\|^p + \|v\|^p)^{1/p} \leq 1 + \eta.$$

(Note that $u \neq 0$, else $\|(x, v)\| = 2^{1/p} > 1 + \varepsilon$.) We have (since $t \mapsto t^p$ is increasing)

$$\|x \pm u\|^p + \|v\|^p \leq (1 + \eta)^p \leq 1^p + \delta = 1 + \delta$$

hence

$$\|x \pm u\|^p \leq 1 + \delta - \|v\|^p = \|u\|^p + \|v\|^p - \|v\|^p + \delta = \|u\|^p + \delta.$$

Taking p -th roots we get

$$\|x \pm u\| \leq \|u\| + \varepsilon$$

since $|\|u\|^p + \delta - \|u\|^p| = \delta$. Let $z = u/\|u\|$. Then

$$\|x \pm z\| \leq \|x \pm u\| + \|z - u\| \leq \|u\| + \varepsilon + 1 - \|u\| = 1 + \varepsilon.$$

(ii). The proof is similar to (i). Indeed, for $\varepsilon_n = \frac{1}{n}$ find the sequence η_n and observe that if a sequence (u_n, v_n) converges weakly to $(0, 0)$ in $X \oplus_p Y$, then u_n converges weakly to 0 in X . \square

We end this section by showing that for finite ℓ_∞ sums we only need to assume that only one of the spaces is LASQ, WASQ or ASQ.

Proposition 5.7. *Let X and Y be nontrivial Banach spaces.*

- (i) $X \oplus_\infty Y$ is LASQ if and only if either X or Y is LASQ.
- (ii) $X \oplus_\infty Y$ is WASQ if and only if either X or Y is WASQ.
- (iii) $X \oplus_\infty Y$ is ASQ if and only if either X or Y is ASQ.

Proof. Let $Z = X \oplus_\infty Y$. We will only prove the ASQ case – the others will follow similarly.

Suppose that Z is ASQ. Let $x_1, x_2, \dots, x_N \in S_X$ and $y_1, y_2, \dots, y_N \in S_Y$. Then (x_i, y_i) is in S_Z for every $i = 1, 2, \dots, N$ and by our assumption there is a sequence $z_n = (u_n, v_n)$ in B_Z such that $\|(x_i, y_i) \pm (u_n, v_n)\| \rightarrow 1$ for every $i = 1, 2, \dots, N$ and $\|z_n\| \rightarrow 1$. Since $\|z_n\| \rightarrow 1$ there is a subsequence such that either $\|u_n\| \rightarrow 1$ or $\|v_n\| \rightarrow 1$. Thus one of the spaces X or Y must be ASQ.

Suppose now that X is ASQ. Let $z_i = (x_i, y_i) \in S_Z$ for $i = 1, 2, \dots, N$ and let $\varepsilon > 0$. We may assume that $x_i \neq 0$ for $i = 1, 2, \dots, N$. Using Proposition 2.1 we can find a $u \in S_X$ $\| \frac{x_i}{\|x_i\|} - u \| \leq 1 + \varepsilon$ for every $i = 1, 2, \dots, N$. Put $z = (u, 0)$. Then

$$\|z_i - z\| \leq \|x_i - u\| = \left\| \|x_i\| \left(\frac{x_i}{\|x_i\|} - u \right) + (1 - \|x_i\|)u \right\| \leq 1 + \varepsilon$$

for every $i = 1, 2, \dots, N$ and Z is ASQ. \square

6. CONNECTION WITH THE IP

In this section we explore the connection between ASQ spaces and the intersection property introduced in [10] (see also [25, Chapter II.4]). For a set $I \subset \mathbb{R}^+$ we will use the notation $B_I = \{x \in X : \|x\| \in I\}$. So for example $B_X = B_{[0,1]}$, $S_X = B_{\{1\}}$ and $B_X \setminus S_X = B_{(0,1)}$.

A Banach space X has the *intersection property* (IP) if for every $\varepsilon > 0$ there exist x_1, x_2, \dots, x_N in X with $\|x_i\| < 1$, $i = 1, 2, \dots, N$, such that if $y \in X$ with $\|x_i - y\| \leq 1$, for every $i = 1, 2, \dots, N$, then $\|y\| \leq \varepsilon$. If X fails the IP, then for some $0 < \varepsilon < 1$ we have $\gamma(\varepsilon) = 1$ where

$$\gamma(\varepsilon) = \sup_{x_1, x_2, \dots, x_n \in B_{(0,1)}} \inf_{y \in B_{(\varepsilon, 1]}} \max_i \|x_i - y\|.$$

We will say that X ε -*fails the IP* if $\gamma(\varepsilon) = 1$. This is very similar to the index $\alpha(\varepsilon)$, $\varepsilon \in [0, 1]$, defined by Maluta and Papini in [36]. Here are two equivalent definitions of $\alpha(\varepsilon)$ (see Proposition 3.3 in [36])

$$\begin{aligned} \alpha(\varepsilon) &= \sup_{x_1, x_2, \dots, x_n \in S_X} \inf_{y \in B_{[\varepsilon, 1]}} \max_i \|x_i - y\| \\ &= \sup_{x_1, x_2, \dots, x_n \in B_{(0,1)}} \inf_{y \in B_{[\varepsilon, 1]}} \max_i \|x_i - y\|. \end{aligned}$$

It is clear that $\alpha(\varepsilon)$ is monotone, $\alpha(0) = 1$, and that for $\varepsilon = 1$ we get the thinness index so that $\alpha(1) = t(X) = 1$ if and only if X is ASQ.

Proposition 6.1. *A Banach space X is ASQ if and only if X ε -fails the IP for all $0 < \varepsilon < 1$.*

Proof. Assume X is ASQ and $0 < \varepsilon < 1$. Since

$$t(X) = \alpha(1) \geq \gamma(\varepsilon) \geq \alpha(\varepsilon) \geq \alpha(0) = 1$$

we get $\gamma(\varepsilon) = 1$.

Conversely. Let $x_1, x_2, \dots, x_N \in S_X$ and $\varepsilon > 0$. Let $z_i = (1 + \varepsilon)^{-1}x_i$. Since X r -fails the IP for $r = 1 - \varepsilon$, there is $y \in B_{(r,1]}$ with $\max_i \|z_i - y\| \leq 1 + \varepsilon$. Then

$$\|x_i - \frac{y}{\|y\|}\| \leq \|x_i - z_i\| + \|z_i - y\| + (1 - \|y\|) \leq 1 + 3\varepsilon.$$

From Proposition 2.1 we conclude that X is ASQ. \square

Remark 6.2. Harmand and Rao, Theorem 1.7 in [24], showed that every Banach space X containing c_0 can be renormed to fail the IP. In Theorem 2.9 we saw that if X is separable it can even be renormed to be ASQ. From Theorem 6.1 we see that this is a strengthening of Harmand and Rao's result for separable spaces.

Example 6.3. For $r > 1$ define

$$X_r = \{f \in C[0, 1] : f(0) = rf(1)\}.$$

X_r is a non-LASQ space which ε -fails the IP for all $\varepsilon \leq 1 - \frac{1}{r}$, but does not ε -fail the IP for any $\varepsilon > 1 - \frac{1}{r}$.

Let $f(x) = (1 - x) + \frac{1}{r}x$. Let $g \in X_r$ with $\|g\| = \varepsilon$. Find $x_0 \in [0, 1]$ such that $|g(x_0)| = \varepsilon$ then

$$\frac{1}{r} + \varepsilon \leq \max_{\pm} |f(x_0) \pm g(x_0)| \leq \|f \pm g\|.$$

With $\varepsilon = 1$ this shows that X_r is not LASQ since $\|f \pm g\|$ is bounded away from 1. It also shows that $\alpha(\varepsilon) \geq \frac{1}{r} + \varepsilon$.

For $f \in X_r$ with $\|f\| < 1$ we have $|f(1)| < \frac{1}{r}$ (if not $|f(0)| \geq 1$).

Let $f_1, f_2, \dots, f_n \in X_r$ with $\|f_i\| < 1$. Find an interval $(a, 1)$ such that $|f_i(x)| < \frac{1}{r}$ for $x \in (a, 1)$. Let $g \in X_r$ with $\text{supp } g \subset (a, 1)$ then $\|f_i + g\| < \frac{1}{r} + \|g\|$ and there exists $\delta > 0$ such that $\|f_i + g\| + \delta \leq \frac{1}{r} + \|g\|$. If we choose g as above with $\|g\| = 1 - \frac{1}{r} + \delta$ then $\max_i \|f_i + g\| \leq 1$ and $\|g\| > 1 - \frac{1}{r}$. Hence X_r $(1 - \frac{1}{r})$ -fails IP.

By the above we also have for $\eta > 0$

$$\gamma((1 - \frac{1}{r}) + \eta) \geq \alpha((1 - \frac{1}{r}) + \eta) \geq \frac{1}{r} + (1 - \frac{1}{r}) + \eta = 1 + \eta > 1.$$

Example 6.4. The space $X = \ell_\infty(C_\Sigma(S^m))$ is ASQ. Here S^m is the Euclidean sphere in \mathbb{R}^{m+1} and

$$C_\Sigma(S^m) = \{f \in C(S^m) : f(s) = -f(-s) \text{ for all } s \in S^m\}.$$

X is not a c_0 -sum of ASQ-spaces nor M-embedded (see [25, Example II.4.6, p. 78]), but a small adjustment to the proof of in [25, Proposition II.4.2 (h), p. 76] shows that X ε -fails the IP for every $0 < \varepsilon < 1$.

Next we will show that every ASQ space contains a separable subspace which is ASQ. The basic idea for the next proof goes back to Theorem 4.4 in Lindenstrauss' memoir [32].

Proposition 6.5. *If X is ASQ, then for every separable subspace Y of X there exists a separable subspace Z with $Y \subset Z \subset X$ and Z is ASQ.*

Proof. Let $Y \subset X$ be separable. With $\varepsilon_n = 2^{-n}$ and $Y_0 = Y$ we construct a sequence of separable subspaces (Y_n) inductively.

Let A_n be a countable dense set in S_{Y_n} . For each finite family G in A_n find y_G in S_X such that $\|x \pm y_G\| < 1 + \varepsilon_n$ for all $x \in G$. Let Y_{n+1} be the closure of $\text{span}\{Y_n, (y_G)\}$. Y_{n+1} is separable since Y_n is.

Define $Z = \overline{\bigcup Y_n}$. Z is separable and ASQ. Let $z_1, z_2, \dots, z_N \in S_Z$ and $\varepsilon > 0$. Choose k such that $\varepsilon_k < \varepsilon/2$ and find x_1, x_2, \dots, x_N in A_k with $\|x_i - z_i\| < \varepsilon/2$. Then there exists a y in $S_{Y_{k+1}} \subset S_Z$ with $\|z_i \pm y\| < 1 + \varepsilon$ for $i = 1, 2, \dots, N$. \square

In [10] Behrends and Harmand asked if dual spaces always have the IP. None of the examples of ASQ spaces we have seen are dual spaces. We ask:

Question 6.6. *Can the dual X^* of a Banach space X be ASQ?*

Remark 6.7. In Remark 2a page 289 in [24] Harmand and Rao noted the following partial answer to the question about the IP: If X^* is such that for any separable subspace Y of X^* there is separable subspace Z with $Y \subset Z \subset X^*$ and Z complemented in X^* , then X^* has the IP. (The assumption is satisfied if e.g. X^* is weakly compactly generated.) Their arguments works also for ASQ spaces and show that an ASQ space can never be a subspace of a weakly compactly generated dual space.

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